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Averaging in quantum stochastics: a soluble model with coloured noise

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Abstract. We calculate the average of the quantum-dynamical evolution operator of a harmonic oscillator linearly coupled to a stochastic field with Lorentzian line shape of arbitrary bandwidth (Ornstein–Uhlenbeck process). In so doing we develop some new techniques to cope with the non-commutativity of the boson operators. We also compare the exact results with van Kampen's second-order cumulant expansion and find that the agreement is good if the bandwidth is large.

1. Introduction

Evolution equations with stochastic coefficients occur in many branches of physics and technology. Noise in electric circuits is one of the oldest examples; see for example Slepian (1958). More recent studies deal with the laser bandwidth problem in quantum optics (Agarwal 1976, Eberly 1976, Avan and Cohen-Tannoudji 1977). However, the noise is always taken to be white, thus δ-correlated, whereas in reality it has a *finite* correlation time, i.e., is coloured and not white. In the case of coloured noise even equations which are linear both in the noise and in the unknown vector ψ,

$$\dot{\psi} = (A + Bg(t) + Cg^*(t))\psi, \tag{1.1}$$

are difficult to solve for the *average* of the vector ψ if the matrices or, more generally, the linear operators A, B, and C do not commute. Precisely this type of problem occurs in quantum optics.

For complex Gaussian coloured noise several *approximate* methods of averaging solutions of equations of the type (1.1) have been put forward. The best known methods are the cumulant expansion (van Kampen 1976) and the matrix or operator continued fraction method (Zoller *et al* 1981). Since, however, the precise range of validity of these methods is not known, it would be valuable to have a non-trivial but exactly soluble model whose solution can be compared with the results obtained by the various approximations. In this paper we want to present such a model.

The model has been taken from quantum mechanics and the simplicity of the (Lie) algebra of harmonic oscillator creation and annihilation operators a and a^+ is used to advantage. We put

$$A = -\mathrm{i}\omega_0 a^+ a, \qquad B = -\mathrm{i}\,\mathrm{e}^{\mathrm{i}\omega t} a^+, \qquad C = -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\omega t} a.$$

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The noise is taken care of by the so-called Ornstein–Uhlenbeck process, which is Gaussian and determined completely by its second-order correlation functions

$$\overline{g(t)g^*(t')} = \exp\{-\gamma|t-t'|\}, \tag{1.2}$$

the odd correlation functions being zero. Here a bar denotes averaging with respect to the stochastic process, which has a Lorentzian power spectrum of width $\gamma > 0$. The evolution equation (1.1) now takes the form of a Schrödinger equation,

$$\mathrm{i}\dot{\psi} = H\psi \tag{1.3}$$

where the Hamiltonian H is given by

$$H = \omega_0 a^+ a + g(t)\,\mathrm{e}^{\mathrm{i}\omega t}a^+ + g^*(t)\,\mathrm{e}^{-\mathrm{i}\omega t}a. \tag{1.4}$$

We are interested in averaging the evolution operator related to (1.3), i.e., the operator $U(t)$ which satisfies the equation

$$\mathrm{i}\,\mathrm{d}U/\mathrm{d}t = HU, \qquad U(0) = \mathbb{1}. \tag{1.5}$$

There are a few other examples of an averaged quantum *dynamics*. One either uses white noise (Alicki and Messer 1982) or exploits the weak coupling limit (Spohn 1977, Dümcke 1981) but in the present context both approaches are not feasible. We note, however, that for the Hamiltonian (1.4) one can obtain the averaged *density matrix* by less involved methods (van Hemmen and Rzążewski 1982).

In § 2 we reduce the problem of averaging the quantum dynamical evolution operator $U(t)$ to a scalar problem which can be solved by evaluating a functional integral. The calculation of this integral, which is done in §§ 3 and 4, requires some non-standard techniques, which turn out to be surprisingly intricate. In view of this intricacy we think that checking the approximate methods with an exact solution is of paramount importance. Some preliminary results are presented in § 5.

2. Reduction to a *c*-number problem

Let us introduce a new stochastic variable,

$$f(t) = g(t)\exp[\mathrm{i}(\omega-\omega_0)t] = g(t)\exp(\mathrm{i}\Omega t), \tag{2.1}$$

where $\Omega = \omega - \omega_0$ is the detuning between the frequencies ω_0 of the oscillator and ω of the field. The new $f(t)$ also represents a complex Gaussian stochastic process. It has mean zero and second-order correlation function

$$\overline{f(t)f^*(t')} = \exp(-\gamma|t-t'|)\exp[\mathrm{i}\Omega(t-t')] \equiv \Delta(t,t'). \tag{2.2}$$

The Hamiltonian (1.4) is now given by

$$H = \omega_0 a^+ a + f(t)\exp(\mathrm{i}\omega_0 t)a^+ + f^*(t)\exp(-\mathrm{i}\omega_0 t)a. \tag{2.3}$$

For the evolution operator in the interaction picture,

$$U_{\mathrm{I}}(t) = \exp(\mathrm{i}\omega_0 t a^+ a)\,U(t) \tag{2.4}$$

we then obtain the equation

$$\mathrm{i}(\mathrm{d}/\mathrm{d}t)\,U_{\mathrm{I}} = [f(t)a^+ + f^*(t)a]U_{\mathrm{I}}, \tag{2.5}$$

where we use the identity

$$\exp(i\omega_0 t a^+ a) a^+ \exp(-i\omega_0 t a^+ a) = \exp(i\omega_0 t) a^+$$

and its Hermitian conjugate. The solution of equation (2.5) can be expressed in normally ordered form (Louisell 1973, § 3.11),

$$U_I(t) = \exp\left(-i\int_0^t d\tau f(\tau) a^+\right) \exp\left(-i\int_0^t d\tau f^*(\tau) a\right) \times \exp\left(-\int_0^t \int_0^t d\tau_1\, d\tau_2\, f^*(\tau_1)\theta(\tau_1-\tau_2) f(\tau_2)\right). \tag{2.6}$$

Here $\theta(\tau) = 1$ if $\tau \geqslant 0$, and zero elsewhere.

To reduce the operator problem to a c-number problem we take the matrix element $U_\alpha(t)$ of $U_I(t)$ with respect to the coherent state $|\alpha\rangle$ (i.e., the state defined by $a|\alpha\rangle = \alpha|\alpha\rangle$),

$$U_\alpha(t) := \langle\alpha|U_I(t)|\alpha\rangle = \exp\left(-\int_0^t \int_0^t d\tau_1\, d\tau_2\, f^*(\tau_1)\theta(\tau_1-\tau_2) f(\tau_2) - i\int_0^t d\tau (f^*(\tau)\alpha + f(\tau)\alpha^*)\right) \tag{2.7}$$

which we write as $U_\alpha[f, f^*]$ to bring out its dependence upon f and f^*. It is known that a bounded operator U is determined completely by its coherent state matrix elements $\langle\alpha|U|\alpha\rangle$. (Klauder and Sudarshan 1968). Applying this observation to $U_I(t)$ we see that we have to evaluate $\langle\alpha|U_I(t)|\alpha\rangle = U_\alpha(t)$ for arbitrary α and then reconstruct $U_I(t)$ from its matrix elements.

Averaging $U_\alpha(t)$ over the complex stochastic process $f(\tau)$ means that we have to perform the functional integral (Simon 1979)

$$\overline{U_\alpha(t)} = \int Df^* Df P[f, f^*] U_\alpha[f, f^*] \tag{2.8}$$

where

$$P[f, f^*] = \mathcal{N} \exp\left(-\int_0^t \int_0^t d\tau_1\, d\tau_2\, f^*(\tau_1) K(\tau_1, \tau_2) f(\tau_2)\right) \tag{2.9}$$

is the 'functional probability distribution' of $f(\tau)$ with normalisation constant $\mathcal{N} = [\det \Delta]^{-1}$ and $K = \Delta^{-1}$. If one discretises the problem, the ensuing formulae are readily understood. Equations (2.7), (2.8), and (2.9) imply that (2.8) is a Gaussian integral which can be performed exactly since the exponent of the integrand is at most quadratic in f. The result is

$$\overline{U_\alpha(t)} = \mathcal{F}(t) \exp[\mathcal{G}(t)\alpha^*\alpha], \tag{2.10}$$

where

$$\mathcal{F}(t) = [\det(\mathbb{1} + V)]^{-1}, \tag{2.11}$$

with $\mathbb{1}$ as the identity operator and $V = \theta\Delta$, which has the integral kernel ($0 \leqslant x, y \leqslant t$)

$$V(x, y) = \int_0^t d\tau\, \theta(x-\tau)\Delta(\tau, y). \tag{2.12}$$

Moreover,

$$\mathcal{G}(t) = -\int_0^t \int_0^t \mathrm{d}\tau_1 \, \mathrm{d}\tau_2 \, B(\tau_1, \tau_2); \tag{2.13}$$

$B(\tau_1, \tau_2)$ is the integral kernel of the operator

$$B = (K + \theta)^{-1} = \Delta(\mathbb{1} + V)^{-1}. \tag{2.14}$$

Using (2.10) we can easily reconstruct the operator $\overline{U_{\mathrm{I}}(t)}$,

$$\overline{U_{\mathrm{I}}(t)} =: \mathcal{F}(t) \exp\{\mathcal{G}(t) a^+ a\} := \mathcal{F}(t) \exp\{\ln(1 + \mathcal{G}(t)) a^+ a\}, \tag{2.15}$$

where : . . . : denotes the normal ordering or the boson operators a^+ and a; a derivation of the second equality may be found, for instance, in Louisell (1973).

3. The determinant of the integral operator $(\mathbb{1} + V)$

Equations (2.11)–(2.15) lead us to the problem of finding the determinant and the inverse of the integral operator $(\mathbb{1} + V)$. The technique we will use to solve this problem is hoped to be of interest in its own right. We, therefore, spell it out in some detail.

The *infinite* determinant $\det(\mathbb{1} + V)$ is defined as the product of the eigenvalues of the operator $\mathbb{1} + V$,

$$\det(\mathbb{1} + V) = \prod_i (1 + \lambda_i). \tag{3.1}$$

Each eigenvalue λ_i of V appears as many times as is required by its algebraic multiplicity, and is determined by $\lambda_i \varphi = V\varphi$ for some $\varphi \neq 0$, i.e.,

$$\lambda_i \varphi(x) = \int_0^t \mathrm{d}y \, V(x, y) \varphi(y). \tag{3.2}$$

Using (2.2) and (2.12) one easily finds an explicit expression for the kernel $V(x, y)$,

$$V(x, y) = -(1/\zeta) \, \mathrm{e}^{-\zeta y} + (1/\zeta + 1/\zeta^*) \theta(x - y) + (1/\zeta) \theta(y - x) \, \mathrm{e}^{\zeta(x-y)} - (1/\zeta^*) \theta(x - y) \, \mathrm{e}^{-\zeta^*(x-y)} \tag{3.3}$$

with $0 \leqslant x, \, y \leqslant t$ and (cf equations (1.2) and (2.1))

$$\zeta = \gamma + \mathrm{i}\Omega, \qquad \zeta^* = \gamma - \mathrm{i}\Omega. \tag{3.4}$$

The integral kernel $V(x, y)$ is continuous in both arguments. Note that $V(0, y) = 0$, whatever y.

The infinite product (3.1) makes sense if $\{\lambda_i\}$ is absolutely summable. This is the case if V is a trace class operator (Simon 1977). Now $V = \theta\Delta$ is indeed trace class since it is defined as the product of two Hilbert–Schmidt operators, θ and Δ, which are both integral operators on $L^2[0, t]$ with bounded and, hence, quadratically integrable kernels (Reed and Simon 1972, § VI.6).

Let us consider the Fredholm integral equation

$$u(x) = \nu \int_0^t V(x, y) u(y) \, \mathrm{d}y; \tag{3.5}$$

t is fixed throughout what follows. There is a one-to-one correspondence between

the characteristic numbers ν and the eigenfunctions $u(x)$ of this equation and the eigenvalues λ and eigenfunctions $\varphi(x)$ of the integral operator V. Moreover we have

$$\det(\mathbb{1}+V)=\prod_i (1+\lambda_i)=\prod_i (1+1/\nu_i) \tag{3.6}$$

with $|\nu_i|\to\infty$ as i increases. Taking advantage of (2.12), (2.2) and (3.4) we differentiate both sides of (3.5) three times with respect to x, so as to find

$$u'(x)=\nu\int_0^x \exp\{-\zeta^*(x-y)\}u(y)\,dy+\nu\int_x^t \exp\{\zeta(x-y)\}u(y)\,dy,$$

or simply

$$u'(x)=\nu I_1+\nu I_2, \tag{3.7a}$$

and

$$u''(x)=-\nu\zeta^* I_1+\nu\zeta I_2, \tag{3.7b}$$

$$u'''(x)=-\nu(\zeta+\zeta^*)u+\nu\zeta^{*2} I_1+\nu\zeta^2 I_2. \tag{3.7c}$$

We then add the equations for u', u'', and u''', and obtain

$$u'''(x)+(\zeta^*-\zeta)u''(x)-|\zeta|^2 u'(x)+\nu(\zeta+\zeta^*)u(x)=0, \tag{3.8}$$

or succinctly, $\mathbb{D}(\nu)u=0$, where $\mathbb{D}(\nu)$ is a differential operator with constant coefficients. So we have the following result: if u is an eigenfunction of the problem (3.5), then u is three times differentiable, satisfies the third-order differential equation (3.8) and, hence, has the form

$$u(x)=\sum_{i=1}^{3} A_i\,e^{a_i x} \tag{3.9}$$

where the a_i are the *roots* (zeros) of the third-order polynomial

$$W_\nu(a)=a^3+(\zeta^*-\zeta)a^2-|\zeta|^2 a+\nu(\zeta+\zeta^*) \tag{3.10a}$$

or, equivalently,

$$W_\nu(a)=a(a-\zeta)(a+\zeta^*)+\nu(\zeta+\zeta^*). \tag{3.10b}$$

To determine the coefficients A_i, $1\leq i\leq 3$, we note that u being a solution to (3.5) implies

$$u(0)=0, \qquad \zeta u'(0)-u''(0)=0, \qquad \zeta^* u'(t)+u''(t)=0, \tag{3.11}$$

as one readily verifies by an explicit calculation invoking (3.7). Combining (3.9) and (3.11) we find a linear homogeneous system of equations for the A_i,

$$\sum_{i=1}^{3} A_i=0, \qquad \sum_{i=1}^{3} a_i(\zeta-a_i)A_i=0, \qquad \sum_{i=1}^{3} e^{a_i t}a_i(\zeta^*+a_i)A_i=0. \tag{3.12}$$

A necessary and sufficient condition for the A_i to be non-zero is that the determinant of the system vanish,

$$D(a_1,a_2,a_3)=\det\begin{pmatrix} 1 & 1 & 1 \\ a_1(\zeta-a_1) & a_2(\zeta-a_2) & a_3(\zeta-a_3) \\ e^{a_1 t}a_1(\zeta^*+a_1) & e^{a_2 t}a_2(\zeta^*+a_2) & e^{a_3 t}a_3(\zeta^*+a_3)\end{pmatrix}=0. \tag{3.13}$$

One may verify that direct substitution of (3.9) into (3.5) leads to the same condition. $D(a_1, a_2, a_3)$ is a function of ν since the a_i are roots of the polynomial W_ν and, thus, depend on ν. The characteristic numbers ν of the integral equation (3.5) are the zeros of the function $D(\nu) = D(a_1, a_2, a_3)$ and, conversely, the zeros of $D(\nu)$ give rise to eigenvalues of the problem (3.5). The a_i always satisfy $W_\nu(a_i) = 0$. Moreover,

$$D(\nu) = \sum_{i=1}^{3} e^{a_i t} q_i a_i (a_i + \zeta^*)^2 \tag{3.14}$$

where

$$q_1 = a_3 - a_2, \qquad q_2 = a_1 - a_3, \qquad q_3 = a_2 - a_1. \tag{3.15}$$

In deriving (3.14) we have used (3.10b) and (3.19). Note the cyclic structure of (3.15).

Denote by M the matrix whose determinant (3.13) has to vanish. A solution to (3.12) is in $\ker(M)$, the kernel of M, and $\dim(\mathrm{ran}\, M)$, the dimension of the range of M, is nothing but its column rank. Since $\dim(\ker M) + \dim(\mathrm{ran}\, M) = 3$ and $\dim(\mathrm{ran}\, M) \geq 1$, we can find at most two independent non-zero solutions of (3.12), i.e., the geometric multiplicity of the eigenvalues ν of (3.5) is at most two. Suppose, in fact, the degeneracy were two. Then $\dim(\mathrm{ran}\, M) = 1$ and all the column vectors of M are to be equal, so that $a_1 = a_2 = a_3$. It is shown in appendix 2 that this can never be the case. Hence the geometric multiplicity of the eigenvalues is one. In appendix 2 we also show: (a) double roots of W_ν do not occur as eigenvalues so that (3.9) with *distinct* a_i is the most general representation of an eigenfunction, and (b) the algebraic multiplicity of the eigenvalues does not exceed one either, i.e., they are *simple*.

$D(\nu)$ is closely related to an entire function. To show this we must make a small detour. $D(\nu)$ as defined in (3.13) is anti-symmetric in a_1, a_2, and a_3, and so is its representation (3.14). Expanding all $\exp(a_i t)$ in (3.15) we find

$$D(\nu) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\sum_{i=1}^{3} a_i^{n+1} q_i (a_i + \zeta^*)^2 \right) \equiv \sum_{n=0}^{\infty} \frac{t^n}{n!} Q_n(a_1, a_2, a_3). \tag{3.16}$$

Because Q_n is an anti-symmetric polynomial in a_1, a_2, and a_3, it is divisible by the anti-symmetric Vandermonde determinant

$$V_3(a_1, a_2, a_3) = \det \begin{pmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{pmatrix} = q_1 q_2 q_3, \tag{3.17}$$

and may be written (Mostowski and Stark 1964, p 341)

$$Q_n(a_1, a_2, a_3) = V_3(a_1, a_2, a_3) S_n(a_1, a_2, a_3), \tag{3.18}$$

where S_n is a *symmetric* polynomial. In fact, S_n itself is a polynomial in three elementary symmetric functions (Mostowski and Stark 1964), which are, according to Vietà's formulae applied to (3.10a), simple analytic expressions in ν:

$$a_1 + a_2 + a_3 = \zeta - \zeta^*, \qquad a_1 a_2 + a_1 a_3 + a_2 a_3 = -|\zeta|^2, \qquad a_1 a_2 a_3 = -\nu(\zeta + \zeta^*). \tag{3.19}$$

Thus $D(\nu)$ may be written as the product of an *entire* function $S(\nu)$,

$$S(\nu) = \sum_{n=0}^{\infty} \frac{t^n}{n!} S_n(\nu), \tag{3.20}$$

and the Vandermonde determinant $V_3(\nu) = q_1 q_2 q_3$; cf (3.17). If ν is an eigenvalue of (3.5), $V_3(\nu)$ never vanishes.

To get a convenient representation of $D(\nu)$ in terms of its zeros, which are in one-to-one correspondence with the ν_i as they appear in (3.6), we prove that $S(\nu)$ is an entire function of order *less than one* (Boas 1954, Titchmarsh 1939). To this end we show that for sufficiently large R there exist constants c_1 and c_2 independent of R such that

$$\max_{|\nu|=R} |D(\nu)| \leqslant c_1 \exp(c_2 R^{1/3}). \tag{3.21}$$

Since V_3 is irrelevant for this estimate, equation (3.21) also holds for $|S(\nu)|$. Indeed, by (3.14)

$$\begin{aligned} |D(\nu)| &= \left| \sum_{i=1}^{3} e^{a_i t} q_i (a_i + \zeta^*)(a_{i+1} - \zeta)(a_{i+2} - \zeta) \right| \\ &\leqslant \sum_{i=1}^{3} |e^{a_i t}||q_i||(a_i + \zeta^*)(a_{i+1} - \zeta)(a_{i+2} - \zeta)|, \end{aligned}$$

and for sufficiently large $|\nu|$ (see appendix 1)

$$|a_i| \leqslant 2|\nu|^{1/3} |\zeta + \zeta^*|^{1/3} \leqslant c|\nu|^{1/3} \tag{3.22}$$

so that

$$\begin{aligned} |D(\nu)| &\leqslant 3 \exp(ct|\nu|^{1/3})(2c|\nu|^{1/3})(c\nu^{1/3} + |\zeta|)^3 \\ &\leqslant c_1 \exp(c_2 |\nu|^{1/3}) \end{aligned} \tag{3.23}$$

for suitable chosen c_1 and c_2. The inequality (3.23) implies together with (3.22) and the product representation $D(\nu) = S(\nu) V_3(\nu)$ that $S(\nu)$ is an entire function whose order does not exceed $\frac{1}{3}$. For this type of function the Hadamard factorisation theorem (Boas 1954, § 2.7, Titchmarsh 1939, § 8.24) asserts that $S(\nu)$ may be written as

$$S(\nu) = S(0) \prod_i (1 - \nu/\nu_i) \tag{3.24}$$

where the ν_i are the zeros of $S(\nu)$, which have infinity as an accumulation point. Then we get for $D(\nu)$

$$D(\nu) = S(0) \prod_i (1 - \nu/\nu_i) V_3(\nu). \tag{3.25}$$

Substituting $\nu = 0$ into equation (3.10) we find the roots $a_1 = 0$, $a_2 = \zeta$, and $a_3 = -\zeta^*$; in addition, $V_3(0) = -|\zeta|^2(\zeta + \zeta^*) \neq 0$ and, by (3.14), $D(0) = |\zeta|^2(\zeta + \zeta^*) \exp(\zeta t)$. Thus, by (3.25),

$$S(0) = -(\zeta + \zeta^*) \exp(\zeta t). \tag{3.26}$$

Each zero ν_i of $D(\nu)$ gives rise to precisely one characteristic value of the Fredholm equation (3.5) and, hence, appears in (3.6). Combining this observation with (3.25) and (3.26) we see that the infinite product (3.6) is given by

$$\prod_i (1 - \nu/\nu_i)|_{\nu=-1} = D(-1)/S(0) V_3(-1), \tag{3.27}$$

and accordingly, using (3.14),

$$\mathcal{F}(t) = \det(\mathbb{1} + V)^{-1} = -\frac{(\zeta + \zeta^*)\exp(\zeta t)\tilde{q}_1\tilde{q}_2\tilde{q}_3}{\sum_{i=1}^{3}\tilde{q}_i\tilde{a}_i(\tilde{a}_i + \zeta^*)^2\exp(\tilde{a}_i t)} \tag{3.28}$$

where the quantities $\tilde{q}_i$ and $\tilde{a}_i$ are to be evaluated at $\nu = -1$, i.e., the $\tilde{a}_i$ are the roots of $W_{-1}(a)$ as defined in (3.10) and $\tilde{q}_1 = \tilde{a}_3 - \tilde{a}_2$, $\tilde{q}_2 = \tilde{a}_1 - \tilde{a}_3$, $\tilde{q}_3 = \tilde{a}_2 - \tilde{a}_1$. This finishes the calculation of the determinant of $(\mathbb{1} + V)$.

4. The inverse of the integral operator $(\mathbb{1} + V)$

Our second problem consists in finding the inverse of the operator $(\mathbb{1} + V)$, i.e., solving the equation

$$(\mathbb{1} + V)G = G(\mathbb{1} + V) = \mathbb{1} \tag{4.1}$$

for the inverse $G = (\mathbb{1} + V)^{-1}$. We will determine its kernel $G(x, y)$ by solving

$$G(x, y) + \int_0^t \mathrm{d}\tau\, V(x, \tau)G(\tau, y) = \delta(x - y). \tag{4.2}$$

The existence of G is secured by first checking $(D - 1) \neq 0$. Differentiating both sides of (4.2) three times with respect to x and recalling (3.7) and (3.8), with $\nu = -1$, we derive the following differential equation for $G(x, y)$,

$$\mathbb{D}(-1)G(x, y) = (\mathrm{d}^3/\mathrm{d}x^3)\delta(x - y) + (\zeta^* - \zeta)(\mathrm{d}^2/\mathrm{d}x^2) - |\zeta|^2(\mathrm{d}/\mathrm{d}x)\delta(x - y), \tag{4.3}$$

where $\mathbb{D}(-1)$ has been defined by means of (3.8) with $\nu = -1$. Note that we differentiate $G(x, y)$ with respect to x. Before proceeding we rewrite (4.3),

$$\mathbb{D}(-1)(G(x, y) - \delta(x - y)) = (\zeta + \zeta^*)\delta(x - y). \tag{4.4}$$

A special solution to (4.4) is given by (cf the last paragraph of the previous section for the notation)

$$\sum_{i=1}^{3}\theta(x - y)\tilde{p}_i\exp\{\tilde{a}_i(x - y)\}, \qquad \tilde{p}_i = (\zeta + \zeta^*)\tilde{q}_i/\tilde{q}_1\tilde{q}_2\tilde{q}_3. \tag{4.5}$$

So the general solution to (4.4) and, hence, (4.3) may be written as

$$G(x, y) - \delta(x - y) = \sum_{i=1}^{3}\exp(\tilde{a}_i x)[B_i(y) + \theta(x - y)\tilde{p}_i\exp(-\tilde{a}_i y)]. \tag{4.6}$$

The $B_i(y)$ are as yet unknown functions of y. For the moment we take y fixed, $0 < y < t$, and write $\psi(x, y) = G(x, y) - \delta(x - y)$. Because of (4.2) the quantity $\psi(x, y)$, which is nothing more than the right-hand side of (4.6), satisfies the same boundary conditions as $u(x)$ in (3.11). Hence we have

$$\begin{pmatrix} 1 & 1 & 1 \\ \tilde{a}_1(\zeta - \tilde{a}_1) & \tilde{a}_2(\zeta - \tilde{a}_2) & \tilde{a}_3(\zeta - \tilde{a}_3) \\ \mathrm{e}^{\tilde{a}_1 t}\tilde{a}_1(\zeta^* + \tilde{a}_1) & \mathrm{e}^{\tilde{a}_2 t}\tilde{a}_2(\zeta^* + \tilde{a}_2) & \mathrm{e}^{\tilde{a}_3 t}\tilde{a}_3(\zeta^* + \tilde{a}_3) \end{pmatrix}\begin{pmatrix} B_1(y) \\ B_2(y) \\ B_3(y) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ -\sum_{i=1}^{3} a_i(\zeta^* + \tilde{a}_i)p_i\,\mathrm{e}^{\tilde{a}_i(t-y)} \end{pmatrix} \tag{4.7}$$

which is to be compared with (3.13). The inhomogeneity stems from (4.5). The fact that the $\tilde{a}_i$ must be evaluated at $\nu = -1$ and $D(-1) \neq 0$ has to be constantly borne in mind. We now solve (4.7) for the $B_i(y)$ and substitute the answer into (4.6). Combining (2.13), (2.14), (2.2), and (4.6) we then get

$$\mathcal{G}(t) = -1 - (\tilde{q}_1\tilde{q}_2\tilde{q}_3)^{-1}\left[(\zeta+\zeta^*)\left(\sum_{i=1}^{3}\tilde{q}_i(\tilde{a}_i+\zeta^*)\,e^{\tilde{a}_i t}\right)^2 \Big/ \sum_{i=1}^{3}\tilde{q}_i\tilde{a}_i(\tilde{a}_i+\zeta^*)^2\,e^{\tilde{a}_i t}\right.$$
$$\left. + \sum_{i=1}^{3}\tilde{q}_i(\tilde{a}_i-\zeta)(\tilde{a}_i+\zeta^*)\,e^{\tilde{a}_i t}\right]. \tag{4.8}$$

This solves the problem of obtaining the exact average (2.15) of the quantum mechanical evolution operator $U_I(t)$.

5. Comparison with the cumulant expansion

The functions $\mathcal{F}(t)$ and $\mathcal{G}(t)$ given by (3.28) and (4.8) may be substituted into (2.15) so as to give an exact expression for $U_I(t)$. We want to compare this expression with the approximate solution which can be obtained by means of van Kampen's cumulant expansion. We incorporate contributions up to second order, assume exact resonance ($\Omega = 0$), and apply equation (12.11) of van Kampen (1976). After some simple algebra we then obtain

$$\overline{U_I(t)} = \mathcal{F}_1(t)\exp\{\ln(1+\mathcal{G}_1(t))a^+a\} \tag{5.1}$$

where

$$\mathcal{F}_1(t) = \exp\left(-\frac{t}{\gamma} - \frac{1}{\gamma^2}(e^{-\gamma t}-1)\right) \tag{5.2}$$

and

$$\mathcal{G}_1(t) = \exp\left(-\frac{2t}{\gamma} - \frac{2}{\gamma^2}(e^{-\gamma t}-1)\right) - 1. \tag{5.3}$$

Note that (5.1) and (2.15) have the same form. Equations (5.2) and (5.3) are to be compared with the exact solutions (3.28) and (4.8). We have done so for various values of γ, with $\Omega = 0$. As seen in figures 1 and 2, for large γ we get a fairly good

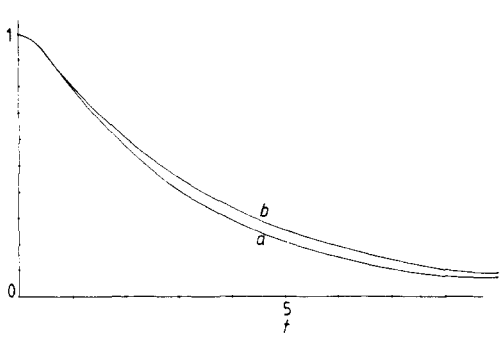

Figure 1. The function $\mathcal{F}(t)$ for $\gamma = 3.0$ and exact resonance ($\Omega = 0$). (*a*) Exact solution (3.28). (*b*) Second-order cumulant expansion result (5.2).

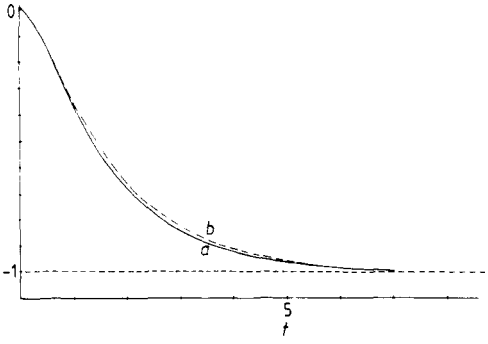

Figure 2. The function $\mathcal{G}(t)$ for $\gamma = 3.0$ and exact resonance ($\Omega = 0$). (*a*) Exact solution (4.8). (*b*) Second-order cumulant expansion result (5.3).

overall agreement. For small γ the agreement is less satisfactory and restricted to small t-values. For instance, as shown in figure 6, $\mathcal{G}_1(t)$ does not reproduce the oscillations which the exact $\mathcal{G}(t)$ exhibits. Qualitatively, $\mathcal{F}_1(t)$ does a slightly better job; see figure 5.

To summarise, as expected ($\tau_c \propto \gamma^{-1}$), the performance of van Kampen's second-order cumulant expansion improves as γ increases. But in view of the approximations involved we think that the agreement for large γ is quite spectacular.

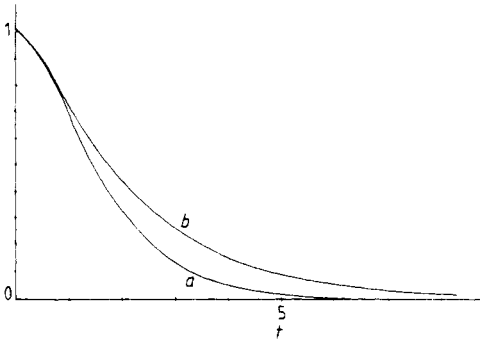

Figure 3. As figure 1, for $\gamma = 1.0$.

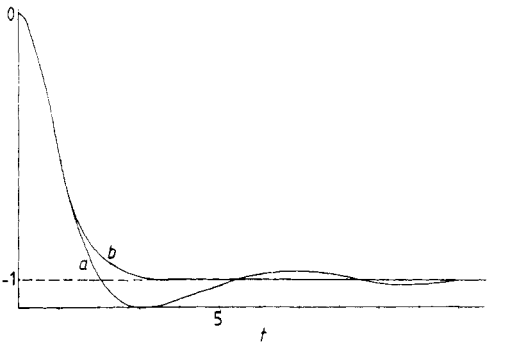

Figure 4. As figure 2, for $\gamma = 1.0$.

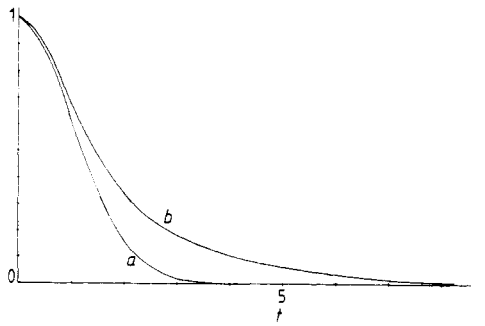

Figure 5. As figure 1, for $\gamma = 0.1$.

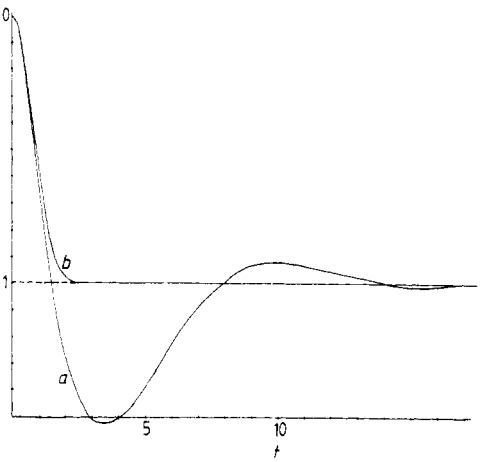

Figure 6. As figure 2, for $\gamma = 0.1$. Note the difference between the solutions.

Acknowledgments

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Appendix 1: A useful inequality

A root x_0 of a polynomial of degree n is a solution to the equation

$$x^n + a_1 x^{n-1} + \ldots + a_n = 0. \tag{A1.1}$$

We wish to estimate $|x_0|$ in terms of the coefficients a_k, $1 \leq k \leq n$. To this end we prove the following lemma.

Lemma. If x_0 is a root of a polynomial of degree n, then

$$|x_0| \leq 2 \max_{1 \leq k \leq n} (|a_k|)^{1/k}. \tag{A1.2}$$

Proof. The inequality (A1.2) is obvious if $x_0 = 0$. Let us therefore assume that $x_0 \neq 0$. Putting

$$\max_{1 \leq k \leq n} (|a_k|)^{1/k} = |a_{k_0}|^{1/k_0} \tag{A1.3}$$

we have

$$|a_k| \leq |a_{k_0}|^{k/k_0}, \qquad 1 \leq k \leq n, \tag{A1.4}$$

and thus

$$|a_k||x_0|^{n-k} \leq |a_{k_0}|^{k/k_0}|x_0|^{n-k}. \tag{A1.5}$$

Rewriting (A1.1),

$$x^n = -(a_1 x^{n-1} + \ldots + a_n), \tag{A1.6}$$

we get, using (A1.5) and the fact that x_0 is a root,

$$|x_0|^n \leq \sum_{k=1}^{n} |a_{k_0}|^{k/k_0}|x_0|^{n-k} \tag{A1.7}$$

so that, because $x_0 \neq 0$,

$$1 \leq \sum_{k=1}^{n} |a_{k_0}|^{k/k_0}|x_0|^{-k} \equiv \sum_{k=1}^{n} q^k \tag{A1.8}$$

where $q = |a_{k_0}|^{1/k}|x_0|^{-1} > 0$. Suppose $q < \frac{1}{2}$. Then

$$\sum_{k=1}^{n} q^k \leq \sum_{k=1}^{\infty} q^k = \frac{q}{1-q} < 1, \tag{A1.9}$$

which contradicts (A1.8). Hence $q \geq \frac{1}{2}$ and

$$|x_0| \leq 2|a_{k_0}|^{1/k} = 2 \max_{1 \leq k \leq n} (|a_k|)^{1/k}. \tag{A1.10}$$

Let us apply the lemma to the third-order polynomial W_ν which has been defined in equation (3.10). For sufficiently large $|\nu|$ we have

$$\max\{|\zeta^* - \zeta|, |\zeta|, (|\nu||\zeta + \zeta^*|)^{1/3}\} = |\zeta + \zeta^*|^{1/3}|\nu|^{1/3}, \tag{A1.11}$$

and thus

$$|a_i(\nu)| \leq c|\nu|^{1/3}, \tag{A1.12}$$

which was to be proved.

Appendix 2: The remaining proofs

We first want to prove that the *geometric* multiplicity of the eigenvalues ν of (3.5) equals one. To this end we turn to the polynomial $W_\nu(a)$ defined in (3.10a) and apply Vietà's formulae (Mostowski and Stark 1964, p 346),

$$a_1a_2a_3 = -\nu(\zeta+\zeta^*), \qquad a_1a_2+a_1a_3+a_2a_3 = -|\zeta|^2, \qquad a_1+a_2+a_3 = -(\zeta^*-\zeta) \tag{A2.1}$$

where $\zeta = \gamma + i\Omega$ and $\gamma > 0$; cf equation (3.4). In the case of a triple root

$$a_1 = a_2 = a_3 = a \tag{A2.2}$$

and

$$a^3 = -\nu(\zeta+\zeta^*), \qquad 3a^2 = -|\zeta|^2, \qquad 3a = \zeta - \zeta^*. \tag{A2.3}$$

Substituting $a = i(2\Omega/3)$ into the expressions for a^2 and a^3 we find

$$a = \pm(2i/\sqrt{3})\gamma, \qquad \Omega = \pm\sqrt{3}\gamma. \tag{A2.4}$$

On the other hand, if we have a triple root (A2.2), the solution to the differential equation (3.8) may be written

$$u(x) = (A_1 + A_2x + A_3x^2)\exp(ax). \tag{A2.5}$$

The function u, whose form *differs* from (3.9), also has to satisfy the boundary conditions (3.11). First, $u(0) = 0$ implies $A_1 = 0$. The remaining two boundary conditions give a homogeneous system of two equations for A_2 and A_3. We find non-trivial solutions only if the determinant vanishes, i.e., if $t > 0$ is such that

$$t^2[|\zeta|^2 + a^2\zeta - 2a\zeta^* - 2a^3] + t[2|\zeta|^2 + 4a(\zeta-\zeta^*) - 6a^2 + 2\zeta^*] + 2(\zeta+\zeta^*) = 0. \tag{A2.6}$$

Combining (A2.4) and (A2.6) one may verify that such a t does not exist. In the case of a double root, where

$$a_1 = a_2 \neq a_3 \tag{A2.7}$$

the solution to (3.8) has the form

$$u(x) = (A_1 + A_2x)\,e^{a_1x} + A_3\,e^{a_3x}, \tag{A2.8}$$

but similar calculations lead to the conclusion that such a solution cannot exist either unless all the A_i vanish. Hence we have shown that the most general representation of the eigenfunctions $u(x)$ is given by (3.9), the geometric multiplicity being one.

Turning to the *algebraic* multiplicity, which occurs in (3.6), let us suppose that we can find a $u \neq 0$ such that

$$(\mathbb{1} - \nu V)(\mathbb{1} - \nu V)u = 0. \tag{A2.9}$$

We have to show

$$v = (\mathbb{1} - \nu V)u = 0 \tag{A2.10}$$

so as to conclude that the eigenvalues ν are *simple*.

Plainly, by (3.7) and (3.8),

$$(\mathbb{1} - \nu V)v = 0 \Rightarrow \mathbb{D}(\nu)v = 0, \tag{A2.11}$$

so that, by (3.9)–(3.13),

$$v(x) = \sum_{i=1}^{3} B_i \, e^{a_i x}, \tag{A2.12}$$

where the B_i are *unique* up to a multiplicative factor. In fact, we have to show $B_i = 0$, $1 \leqslant i \leqslant 3$. By the definition of v,

$$(\mathbb{1} - \nu V)u = \Sigma_i B_i \, e^{a_i x}, \tag{A2.13}$$

and repeating the arguments which lead from (3.7) to (3.8) we get

$$\mathbb{D}(\nu)u = \Sigma_i B_i [W_\nu(a_i) - \nu(\zeta + \zeta^*)] \, e^{a_i x},$$

i.e.,

$$\mathbb{D}(\nu)u = -\nu(\zeta + \zeta^*)\Sigma_i B_i \, e^{a_i x}, \tag{A2.14}$$

since $W_\nu(a_i) = 0$. Equation (A2.14) may be solved and one finds

$$u(x) = \nu(\zeta + \zeta^*)\Sigma_i \frac{B_i q_i}{q_1 q_2 q_3} \, x \, e^{a_i x}. \tag{A2.15}$$

However, returning to (A2.9), we see that

$$u = 2\nu V u - \nu^2 V(Vu) \equiv 2\nu V u - \nu^2 V w \tag{A2.16}$$

and, whatever w, u is bound to satisfy the boundary conditions (3.11). We now have to check whether (A2.15) is consistent with these boundary conditions, i.e.,

$$\zeta u'(0) = u''(0) = 0 \Rightarrow \Sigma_i B_i q_i [\zeta - 2a_i] = 0, \tag{A2.17a}$$

$$\zeta^* u'(t) + u''(t) = 0 \Rightarrow \Sigma_i B_i \, e^{a_i t} q_i [\zeta^*(a_i t + 1) + a_i(2 + a_i t)] = 0. \tag{A2.17b}$$

Moreover, we always have (3.12), including $\Sigma_i B_i = 0$. The equations (A2.17) and (3.12) are incompatible unless all B_i vanish.

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